Strongly reduced gap in the zigzag spin chain with a ferromagnetic interchain coupling

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We study a spin 1/2 Heisenberg zigzag spin chain model near decoupled two chains. Taking into account a symmetry breaking perturbation, we discuss the existence of an energy gap in the ferromagnetic interchain coupling as well as the antiferromagnetic one. In the ferromagnetic model, a marginally relevant fixed line reduces the gap strongly, so that the correlation length becomes an astronomical length scale even in order 1 coupling. This result agrees with density matrix renormalization group results.

I. INTRODUCTION

It is interesting to study the effects of a frustration in a quantum system. The zigzag spin chain is one of the simplest quantum spin models with a frustration. An experiment on the compound $SrCuO_2$ reports a remarkable suppression of the three dimensional ordering temperature due to a strong quantum fluctuation enhanced by the frustration[1]. Here we study the effect of frustration in the zigzag chain with the following Hamiltonian

$$H = \sum_i \left[J_2(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1}) + J_1 \mathbf{S}_i \cdot (\mathbf{T}_i + \mathbf{T}_{i+1}) \right]. \tag{1}$$

The operators $\mathbf{S}_i$ and $\mathbf{T}_i$ are $s = 1/2$ spins. Here we treat an interchain coupling J_1 as a perturbation to the two decoupled antiferromagnetic ($J_2 > 0$) spin chains.

It is well-known that the zigzag chain model with antiferromagnetic region $J_1 > 0$ has a dimer phase and gapless phase[2]. The region of the dimer phase[3] ($0 < J_1 < J_c \sim 4.15 J_2$) contains a Majumdar Ghosh point $J_1 = 2J_2$ where the dimerized ground state is obtained exactly[4]. The point $J_1 = 0$ is a critical point of the two decoupled antiferromagnetic chains. There is another critical point $J_1 = -4J_2$, where the level crossing between singlet and fully polarized ferromagnetic ground states occurs. There is an exact solution[5] of a fully polarized ferromagnetic ground state in $J_1 < -4J_2$, and for a doubly degenerated ground state at the critical point $J_1 = -4J_2$. A numerical analysis in a region $J_1 > -4J_2$ indicates a complicated size dependence of the ground state energy[6,7].

It has been long believed that this model is gapless[8,9] for a small ferromagnetic region $J_1 < 0$. In several papers however, one loop renormalization group (RG) shows an instability of the critical point $J_1 = 0$ both in ferromagnetic ($J_1 < 0$) and antiferromagnetic ($J_1 > 0$) region due to a Lorentz symmetry breaking perturbation[10,11]. This fact is a puzzling because the expected energy gap produced by the unstable flow has never been observed in the ferromagnetic region either numerically or experimentally. If it were gapless, there should be a new stable critical point missed in the one loop approximation. In this paper, we clarify a natural mechanism to solve this problem. We conclude that the actual energy gap is finite but very tiny in an extended region of the coupling constant space. We can understand unstable the RG flow and numerical analysis consistently. The energy gap is too small to observe in any numerical method.

A stable large scale reduction is an important and difficult fundamental problem in theoretical physics. In particle physics, any unified theory has this kind of hierarchy problem. Generally speaking, a unified theory has only one large energy scale, and also should explain the generation of a small energy scale to describe the low energy physics. These two requirements makes unified theories quite unnatural, since we need a fine tuning of the coupling constants for large scale reduction. We believe that nature do not chose special value of the coupling constants. From the view point of hierarchy problem in the theoretical physics, this zigzag chain model can be an interesting example in statistical physics. We conclude that the very tiny gap is always obtained in an extended region of the ferromagnetic interchain coupling unlike the antiferromagnetic coupling.

This paper is organized as follows. In section II we construct an effective field theory for the zigzag chain and calculate the beta functions. In section III, we study the RG flow and discuss an energy gap reduced strongly in the ferromagnetic interchain coupling region. In section IV, we calculate the energy spectrum to describe the results of numerical calculation. In section V, we show our numerical analysis.

II. EFFECTIVE FIELD THEORY

The unperturbed theory is two decoupled Heisenberg antiferromagnetic chains whose effective theory is two decoupled $SU(2)_1$ WZW models[10]. We introduce two free bosons $\varphi_a(z,\bar{z}) = \varphi_a(z) + \bar{\varphi}_a(\bar{z})$ ($a = 1, 2$) with

$$S_0 = \frac{1}{2\pi} \int d^2 z \left(\partial\varphi_1 \bar{\partial}\varphi_1 + \partial\varphi_2 \bar{\partial}\varphi_2 \right). \tag{2}$$

with two point functions $\langle \varphi_a(z)\varphi_b(0)\rangle = -\delta_{ab}\log z$, $\langle \bar{\varphi}_a(\bar{z})\bar{\varphi}_b(0)\rangle = -\delta_{ab}\log \bar{z}$, for two WZW models. The spin operator is written in terms of two bosons

$$2\pi\ \mathbf{S}_j = \mathbf{J}_1 + \bar{\mathbf{J}}_1 + (-1)^j M tr[(g_1 + g_1^\dagger)\frac{{}^t\boldsymbol{\sigma}}{2}], \tag{3}$$

$$2\pi\ \mathbf{T}_j = \mathbf{J}_2 + \bar{\mathbf{J}}_2 + (-1)^j M tr[(g_2 + g_2^\dagger)\frac{{}^t\boldsymbol{\sigma}}{2}], \tag{4}$$

where M is a nonuniversal real constant. Here, the SU(2) currents and primaries are written in two bosons and Klein factors

$$J_a^+(z) = \eta_{a\uparrow}\eta_{a\downarrow}e^{i\sqrt{2}\varphi_a(z)}, \quad J_a^z(z) = \frac{i}{\sqrt{2}}\partial\varphi_a(z), \tag{5}$$

$$\bar{J}_a^+(z) = \bar{\eta}_{a\uparrow}\bar{\eta}_{a\downarrow}e^{-i\sqrt{2}\bar{\varphi}_a(\bar{z})}, \quad \bar{J}_a^z(\bar{z}) = \frac{-i}{\sqrt{2}}\bar{\partial}\bar{\varphi}_a(\bar{z}),$$

$$g_{a\uparrow\uparrow} = \eta_{a\uparrow}\bar{\eta}_{a\uparrow}e^{i(\varphi_a+\bar{\varphi}_a)/\sqrt{2}}, \quad g_{a\uparrow\downarrow} = \eta_{a\uparrow}\bar{\eta}_{a\downarrow}e^{i(\varphi_a-\bar{\varphi}_a)/\sqrt{2}},$$

$$g_{a\downarrow\uparrow} = \eta_{a\downarrow}\bar{\eta}_{a\uparrow}e^{-i(\varphi_a-\bar{\varphi}_a)/\sqrt{2}}, \quad g_{a\downarrow\downarrow} = \eta_{a\downarrow}\bar{\eta}_{a\downarrow}e^{-i(\varphi_a+\bar{\varphi}_a)/\sqrt{2}},$$

with $a = 1, 2$. The Klein factors obey anticommutation relation $\{\eta_{a\alpha}, \eta_{b\beta}\} = 2\delta_{ab}\delta_{\alpha\beta}$, to satisfy the following correct operator product expansion for the SU(2) symmetry

$$J_a^k(z)g_{b\alpha\beta}(w,\bar{w}) \sim \frac{\delta_{ab}/2}{z-w}({}^t\sigma^k g_a)_{\alpha\beta}(w,\bar{w}),$$

$$\bar{J}_a^k(\bar{z})g_{b\alpha\beta}(w,\bar{w}) \sim -\frac{\delta_{ab}/2}{\bar{z}-\bar{w}}(g_a {}^t\sigma^k)_{\alpha\beta}(w,\bar{w}). \tag{6}$$

The interaction operators

$$S_{int} = \int \frac{d^2z}{2\pi}\sum_{i=1}^{5}\lambda_i\phi_i(z,\bar{z}), \tag{7}$$

can be represented in terms of two $SU(2)$ currents and primary fields of the WZW model

$$\begin{aligned}
\phi_1(z,\bar{z}) &= \mathbf{J}_1(z)\cdot\bar{\mathbf{J}}_1(\bar{z}) + \mathbf{J}_2(z)\cdot\bar{\mathbf{J}}_2(\bar{z}),\\
\phi_2(z,\bar{z}) &= \mathbf{J}_1(z)\cdot\bar{\mathbf{J}}_2(\bar{z}) + \mathbf{J}_2(z)\cdot\bar{\mathbf{J}}_1(\bar{z}),\\
\phi_3(z,\bar{z}) &= tr[g_1 i(\partial-\bar{\partial})g_2],\\
\phi_4(z,\bar{z}) &= trg_1 i(\partial-\bar{\partial})trg_2,\\
\phi_5(z,\bar{z}) &= \mathbf{J}_1(z)\cdot\mathbf{J}_2(z) + \bar{\mathbf{J}}_1(\bar{z})\cdot\bar{\mathbf{J}}_2(\bar{z}),
\end{aligned} \tag{8}$$

where $\lambda_2 = J_1/\pi$, $\lambda_3 = -J_1M^2/\pi$, $\lambda_4 = J_1M^2/(2\pi)$, $\lambda_5 = J_1/(2\pi)$ and λ_1 is a certain negative constant. The initial coupling constant λ_1 is roughly estimated as the order 1/10 in numerical analysis for single linear chain[12]. Note that the the perturbations $\phi_3(z,\bar{z})$ and $\phi_4(z,\bar{z})$ include operators with the conformal dimension $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{3}{2})$, which break the Lorentz and the parity symmetry[10,11]. The original spin model has SU(2) symmetry, translational symmetry and a permutation with a translation of the one chain

$$\mathbf{S}_i \to \mathbf{T}_{i+1}, \quad \mathbf{T}_i \to \mathbf{S}_i, \tag{9}$$

which forbids interactions with dimension $x < 2$. If this symmetry is broken, the symmetry breaking can be measured by the following order parameter

$$\mathbf{S}_i\cdot(\mathbf{T}_i - \mathbf{T}_{i+1}) \sim tr[(g_1+g_1^\dagger)\frac{{}^t\boldsymbol{\sigma}}{2}]\cdot tr[(g_2+g_2^\dagger)\frac{{}^t\boldsymbol{\sigma}}{2}]. \tag{10}$$

This order parameter also changes sign under a translation of one spin $\mathbf{T}_i \to \mathbf{T}_{i+1}$. The operator product expansion

$$\phi_i(z,\bar{z})\phi_j(0,0) \sim \sum_k \frac{C_{ijk}}{|z|^2}\phi_k(0,0),$$

gives one loop renormalization group beta function

$$-l\frac{d\lambda_k}{dl} = \frac{1}{2}\sum_{ij}C_{ijk}\lambda_i\lambda_j,$$

which has the following practical form

$$\begin{aligned}
l\frac{d\lambda_1}{dl} &= \lambda_1^2 - \lambda_3\lambda_4 - \lambda_4^2,\\
l\frac{d\lambda_2}{dl} &= \lambda_2^2 + \lambda_3\lambda_4 + \lambda_3^2,\\
l\frac{d\lambda_3}{dl} &= -\frac{1}{2}\lambda_1\lambda_3 + \frac{3}{2}\lambda_2\lambda_3 + \lambda_2\lambda_4,\\
l\frac{d\lambda_4}{dl} &= \lambda_1\lambda_3 + \frac{3}{2}\lambda_1\lambda_4 - \frac{1}{2}\lambda_2\lambda_4,\\
l\frac{d\lambda_5}{dl} &= \frac{1}{2}\lambda_3\lambda_4.
\end{aligned} \tag{11}$$

Note that the beta functions have a symmetry $\lambda_3 \to -\lambda_3$, $\lambda_4 \to -\lambda_4$, which corresponds to a translation on one chain $\mathbf{T}_i \to \mathbf{T}_{i+1}$ in the original spin model.

III. RENORMALIZATION GROUP FLOW

The critical point $J_1 = 0$ divides ferromagnetic and antiferromagnetic dimer phases. Both regions have energy gap. The renormalization flow diverges eventually along a stable direction $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (0,1,\frac{1}{\sqrt{2}},0)$ for the ferromagnetic coupling $J_1 < 0$, and along another stable direction $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (0,1,\frac{-1}{\sqrt{2}},0)$ for the antiferromagnetic coupling $J_1 > 0$. The current-current coupling λ_2 grows with positive value both in ferromagnetic and antiferromagnetic region. In the effective field theory, the ferromagnetic model does not differ from the antiferromagnetic model transformed by $\mathbf{T}_i \to \mathbf{T}_{i+1}$, which changes the sign of λ_3 and λ_4. This fact suggests that both ferromagnetic and antiferromagnetic models have the same dimerization pattern. The correlation length ξ behaves as

$$\xi \sim a\exp c|J_1|^{-\tilde{\nu}} \tag{12}$$

with $\tilde{\nu} = 2/3$ for $J_1 \sim 0$ and the lattice spacing a. This result is obtained in another RG method for the RG equation[13]. In the antiferromagnetic region, this scaling formula of the correlation length holds for relatively

large value in $0 < J_1 < J_c \sim 4.15 J_2$, even though the formula is obtained merely by one loop approximation. We demonstrate it by numerical results from density matrix renormalization group calculation(DMRG)[8] in Fig. 1. The gap scaling formula can fit the numerical result quite well. This agreement even in relatively strong coupling region is not so surprising. There are many systems with Kosterlitz Thouless type phase transition, where the gap scaling formula calculated in the weak coupling region holds even in the strong coupling region. In the ferromagnetic region, however, this scaling formula does not hold for $|J_1| \sim J_2$. The energy gap in this region cannot be found in any numerical analysis. There should be some special reason.

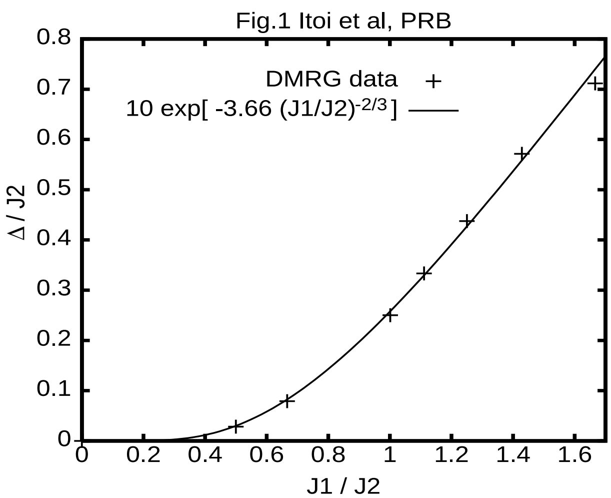

FIG. 1. The energy gap Δ vs. the antiferromagnetic interchain coupling $J_1 > 0$ is depicted. The solid line is drawn by the scaling formula $\Delta \sim \exp(-c|J_1|^{-2/3})$ for small J_1 in the infinite order phase transition, using Eq.(12) with $\xi \sim 1/\Delta$. The gap Δ calculated by the DMRG is obtained from Ref.[8].

We clarify in the following why the gap scaling formula in ferromagnetic region differs from the antiferromagnetic region. The beta functions for λ_1, λ_2, λ_3, λ_4 have simultaneous zero on a line

$$\lambda_1 = \lambda_2 = 0, \ \lambda_3 + \lambda_4 = 0. \tag{13}$$

This line behaves like a fixed line for a four dimensional coupling constant space $(\lambda_1, \cdots, \lambda_4)$, since λ_5 does not enter their beta functions. Near this line, we define the deviation of the running coupling constants from the line by $\lambda_3 = \lambda + \delta\lambda_3$, $\lambda_4 = -\lambda + \delta\lambda_4$. The linearized beta functions are

$$l\frac{d}{dl}(\delta\lambda_1 - \delta\lambda_2) \sim 0,$$
$$l\frac{d}{dl}(\delta\lambda_1 + \delta\lambda_2) \sim 2\lambda(\delta\lambda_3 + \delta\lambda_4),$$
$$l\frac{d}{dl}(\delta\lambda_3 + \delta\lambda_4) \sim -\lambda(\delta\lambda_1 - \delta\lambda_2),$$
$$l\frac{d}{dl}(\delta\lambda_3 - \delta\lambda_4) \sim 0. \tag{14}$$

The eigenvalues of the scaling matrix all vanish on that line. This marginal property of the fixed line yields a remarkable phenomenon. The linearized flow near the fixed line can be integrated as follows

$$\begin{aligned}
&\delta\lambda_1(l) - \delta\lambda_2(l) \sim A,\\
&\delta\lambda_1(l) + \delta\lambda_2(l) \sim C + 2\lambda B \ln l\\
&\delta\lambda_3(l) + \delta\lambda_4(l) \sim B - \lambda A \ln l\\
&\delta\lambda_3(l) - \delta\lambda_4(l) \sim D,
\end{aligned} \tag{15}$$

where A, B, C and D are integration constants determined by initial coupling constants which are nonuniversal. If the initial coupling constants lie near this fixed line, $A + C$ is negative, $A - C \sim -2J_1/\pi$, $B + D = -J_1 M^2/\pi - \lambda$ and $B - D = J_1 M^2/(2\pi) - \lambda$. We can study the nature of the RG flow numerically together with an analytic argument based on the behavior of the fixed line. Although the line is unstable except the case $A = B = 0$, there is an extended region where the running couplings $\delta\lambda_1(l) + \delta\lambda_2(l)$ and $\delta\lambda_3(l) + \delta\lambda_4(l)$ flow toward 0. In this case other couplings are renormalized logarithmically, and the running coupling constants spend a long time (long length scale) near the fixed line (13).

In the ferromagnetic region with small $|J_1|$, we have $A < 0$, $B > 0$, $C < 0$, $\lambda > 0$. In this case, $\delta\lambda_3(l) + \delta\lambda_4(l)$ does not flow toward 0, and then the flow is free from the fixed line as well as in the antiferromagnetic region. In this region, we consider the gap scaling formula (12) holds and the gap is still small enough. At $J_1 \sim -0.1 J_2$ however, the constant A changes sign and the running coupling constants start to flow toward 0. The correlation length grows again by the effect of the fixed line (13). The correlation length can be estimated approximately from the length scale l where the running coupling constant $\lambda_i(l)$ diverges. The numerical solution of the RG equation shows a dependence of the correlation length on the non-universal constant M. Typically, the minimal correlation length in the ferromagnetic region becomes an astronomical length scale more than $10^{36}a$ for $J_1 \sim -0.2 J_2$ and $M = 1$. The interaction between two chains may change the non-universal quantities λ_1 and M^2. However, the qualitative nature of the flow is unchanged for $M^2 < 2$ and $J_1 < 0$. If $M^2 > 2.5$, the gap may be observed by numerical analysis. We estimate the gap of the zigzag chain by numerically evaluating the renormalization group equations. The data $\ln\Delta$ vs. J_1 is plotted in Fig. 2 setting the nonuniversal quantities $M = 1$ and $\lambda_1 = -0.24/\pi$ as typical values. Also in this case we could observe asymmetric property between the ferromagnetic and antiferromagnetic coupling J_1. The theoretical fitting function with $\tilde{\nu} = 2/3$ can fit the data in the antiferromagnetic side $J_1 > 0$ better than those in the ferromagnetic side $J_1 < 0$.

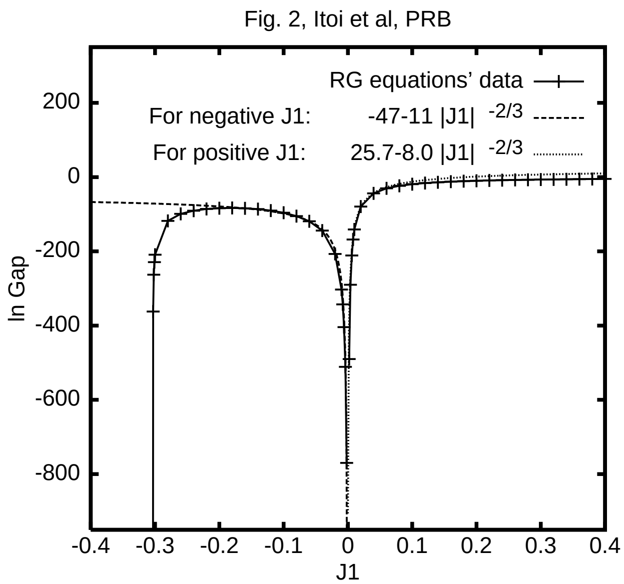

FIG. 2. The logarithm of gap $\ln\Delta$ vs. small J_1 estimated by the scale where the solution of the RG equation has the singularity. The fitting line is obtained by fitting formula $\Delta = a\exp(-c/|J_1|^{2/3})$ with data close to zero. Gaps obtained from RG equations in between $0.1 > J_1 > -0.1$ is used in fitting.

This fact implies a quite unusual phenomenon. The infinite system differs essentially from a macroscopic system with the finite size. A system with a macroscopic finite size is described in a massless theory on the fixed line, while an infinite system is described in a massive theory. We can employ the massless field theory for a macroscopic system available in an ordinary condensed matter experiment, since the correlation length becomes an astronomical length scale. We can understand the quite slow convergence in numerical methods in this region. Although the beta functions depend on the renormalization scheme[10,11], the existence of the marginally relevant fixed line is scheme independent and the strong gap reduction in an extended region occurs universally. On the other hand, in the antiferromagnetic interchain coupling, the flow is free from the fixed line and the correlation length becomes $\xi \sim 7a$ for $J_1 \sim J_2$. This result is consistent with the numerical analysis in Ref.[8] as depicted in Fig. 1.

IV. LOW ENERGY LEVELS

Next we consider the effect of the chiral operators $\phi_5(z) = \mathbf{J}_1(z)\cdot\mathbf{J}_2(z)$ and $\bar\phi_5(\bar z) = \bar{\mathbf{J}}_1(\bar z)\cdot\bar{\mathbf{J}}_2(\bar z)$ with conformal dimension $(2,0)$ and $(0,2)$ respectively. In an ordinary macroscopic system size, we can employ the massless theory as a good effective theory. A real fermion representation is useful to see the effect of the chiral operator[9,10,14]

$$\xi_1(z) + i\xi_2(z) = \sqrt{\frac{2}{a}}\eta_+ \exp\left[-i\frac{\varphi_1(z)+\varphi_2(z)}{\sqrt{2}}\right]$$
$$\xi_3(z) + i\xi_4(z) = \sqrt{\frac{2}{a}}\eta_- \exp\left[-i\frac{\varphi_1(z)-\varphi_2(z)}{\sqrt{2}}\right]. \quad (16)$$

Here $\eta_\pm$ are written in a bilinear of $\eta_{a\alpha}$ such that the total current for the SO(3) generator becomes $\mathbf{J}_1+\mathbf{J}_2 = \frac{i}{2}\boldsymbol{\xi}\times\boldsymbol{\xi}$, where $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$. Then the fields $\boldsymbol{\xi}$ and ξ_4 correspond to a triplet and singlet excitation, respectively. In this mapping, the chiral perturbation operator has a free field representation

$$\phi_5(z) = -\frac{1}{2}\partial\varphi_1\partial\varphi_2 + \eta_{1\uparrow}\eta_{1\downarrow}\eta_{2\downarrow}\eta_{2\uparrow}\cos[\sqrt{2}(\varphi_1(z)-\varphi_2(z))]$$
$$= \frac{1}{4}\boldsymbol{\xi}\cdot\partial\boldsymbol{\xi} - \frac{3}{4}\xi_4\partial\xi_4. \quad (17)$$

The chiral perturbation operator (17) makes the shift of the spin wave velocities of triplet and singlet fields[9]

$$v_t = v + \frac{J_1}{4\pi}, \quad v_s = v - \frac{3J_1}{4\pi}. \quad (18)$$

These shifts are found in the finite size correction to the low energy levels of two triplet boundary operators $\xi_j\xi_k$, $\xi_i\xi_4$ with $j,k = 1,2,3$ in open boundary condition(OBC) and two triplet operators $\sigma_i\sigma_j\mu_k\mu_4$, $\sigma_i\mu_j\mu_k\sigma_4$ for periodic boundary condition (PBC)[9]. Each gap in the leading finite size correction in OBC depends on J_1

$$\Delta E_{jk} \sim \frac{\pi v_t}{L} = \frac{\pi(v+J_1/4\pi)}{L},$$
$$\Delta E_j \sim \frac{\pi(v_t+v_s)}{2L} = \frac{\pi(v-J_1/4\pi)}{L}. \quad (19)$$

In PBC, all belong to the same energy independent of J_1

$$\Delta E \sim \frac{2\pi(3v_t+v_s)}{4L} = \frac{2\pi v}{L}. \quad (20)$$

There are some logarithmic corrections for a short chain mainly through $\lambda_1(L)+\lambda_2(L) \sim 2/\ln L$ much larger than the effect of renormalization $\delta\lambda_5(L) \sim 6\times10^{-4}\ln L$. We can expect the flat band instability at $J_1 = -4\pi v$ in the fermionization. This instability is a sign of a level crossing between the singlet and the ferromagnetic states.

V. NUMERICAL ANALYSIS

These arguments are consistent with DMRG analysis on the OBC zigzag chain. Here we chose $J_2 = 1$ and calculate the low energy levels numerically by DMRG method[15]. The gap is too small and the correlation length is too large to check the energy gap directly for $J_1 < 0$ in numerical analysis. However, the results obtained in previous sections can be supported by the lowest three energy levels of the system. The ground state and first excited state for each chain at $J_1 = 0$ are unique

and of total spin zero and one respectively[16] for even length chains. We label the ground state for the zigzag chain as $(0,0)$ which is the product of the ground states of the two chains. The two degenerate first excited states of total spin one, $(0,1)^t$ and $(0,1)^s$, are the parity odd and even states as the product of the ground state and the first excited state from different chains. The parity is for the symmetry of S chain and T chain permutation.

When J_1 decreases from zero, the excitation energy behaves as gapless for the infinite system in our finite size scaling fitting. The energy of $(0,1)^t$, $\pi v_t/L$ decreases, and the one $\pi(v_s+v_t)/2L$ of $(0,1)^s$ increases for small $|J_1|$. In Fig. 3, we demonstrate the two different spin velocities obtained by scaling for an arbitrary point $J_1=-0.5$ in the phase with small and negative J_1. We have kept $m=500$ states in DMRG calculation for OBC chains, and the biggest truncation error in DMRG is 10^{-7}.

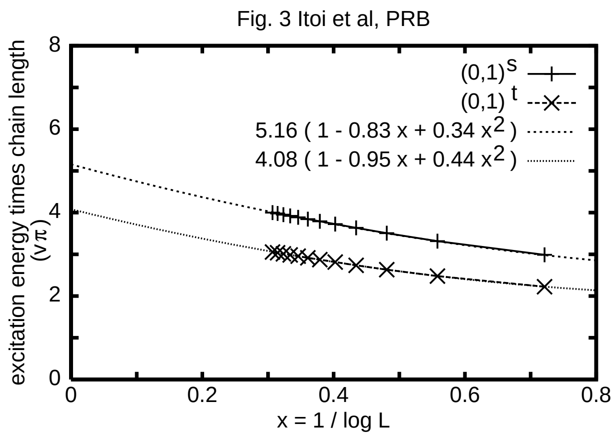

FIG. 3. The spin velocities for $J_1=-0.5$. The excitation energies calculated by DMRG for the state $(0,1)^t$ and $(0,1)^s$ multiplied by L is plotted vs. $1/\ln L$ with L up to 48. Two different spin velocity v_t and v_s are obtained for them respectively by least square fitting. The system behaves as its gap scaling being $1/L$.

The linear increasing of v_t and decreasing of v_s predicted in Eq.(18) for $J_1\sim 0$ can be observed in Fig. 4 by the low energy levels for OBC $L=8$ chains. In Fig. 4, DMRG result is exact for such a short length $L=8$. We have plotted the lowest five energy levels for each S_z^{total} in the figure vs. the interchain coupling J_1.

In Fig. 5, the low energy levels obtained by exact diagonalization for a PBC $L=6$ chain is shown. We can check in the figure that the two lowest excitation energies are independent of J_1, as we expected in Eq.(20). The two lowest excitation energies are of total spin 1. In the figure, eight lowest energy levels for each total momentum and S_z^{total} of $L=6$ (12 sites) PBC chains has been plotted vs. interchain coupling J_1.

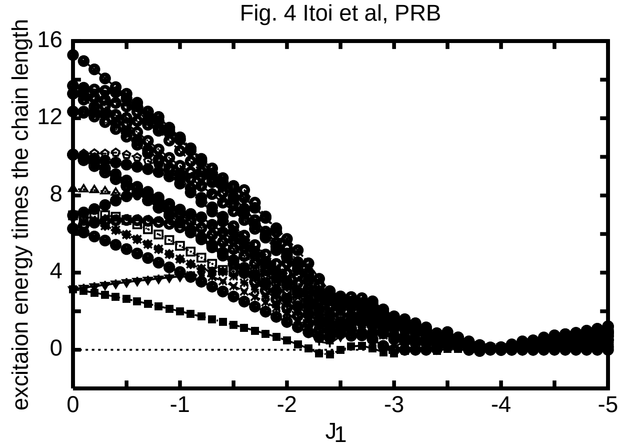

FIG. 4. Low excitation energies calculated by DMRG for zigzag open chain for length $L=8$ with negative J_1. The lowest excited state at $J_1\sim 0$ is $(0,1)^t$ and the second lowest is $(0,1)^s$. The states collapse to zero energy when J_1 approaches -4. A non singlet ground state seems to appear at $J_1\sim -2$.

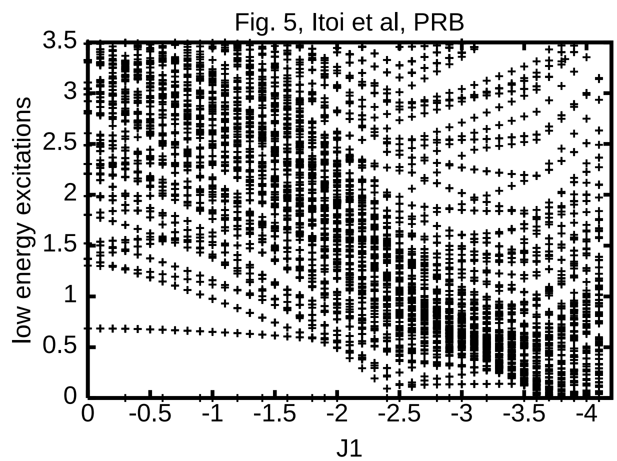

FIG. 5. Low excitation energies calculated by exact-diagonalization for zigzag PBC chains with $L=6$ (12 sites). For each negative J_1, eight lowest energy levels for each total momentum and S_z^{total} is plotted. The two lowest excited states at $J_1\sim 0$ are of total spin 1. Their energy is degenerate and almost independent of J_1 as predicted in Eq.(20).

The low energy excitations are described in the critical theory near $J_1=0$ with the shifted spin wave velocities, therefore we can naturally exclude a different critical theory at a new fixed point away from $J_1=0$ which describes this system.

In Fig. 4 flat band instability is demonstrated. It occurs at $J_1=-4$ and the system enters the ferromagnetic phase with the maximum spin $S^{tot}=L$ ground state, which is consistent with previously obtained results[5,6]. A non singlet ground state seems to occur near $J_1=-2$ as pointed by Cabra et al[11]. It is difficult to judge whether this partially polarized ground state can survive in the infinite length limit. In a macroscopic system size, it

seems alive in $J_1 > -4$.

VI. DISCUSSIONS

In conclusion, the divergent RG flow by the Lorentz symmetry breaking perturbation certainly produces an energy gap in the zigzag chain with the ferromagnetic interchain coupling as well as in the antiferromagnetic model. In an extended region of the ferromagnetic interchain coupling with order 1, the correlation length can be extremely large due to the effect of the marginally relevant fixed line. If there is a finite energy gap, it becomes too tiny to observe. The quantum fluctuation enhanced by the frustration yields the extraordinary reduction of the energy gap, which cannot be checked in ordinary macroscopic physics. The slow convergence in numerical analysis can be understood for this reason. The DMRG analysis shows that the massless theory near the two decoupled chains describes well the low energy levels in a macroscopic system.

Here, we comment on an experimental result on the zigzag chain compound $SrCuO_2$ and the linear chain compound Sr_2CuO_3. Motoyama, Eisaki and Uchida show that the temperature dependence of the spin susceptibility of the zigzag chain $SrCuO_2$ differs from that in the linear chain Sr_2CuO_3[17]. The drastic decreasing of the susceptibility is observed in the linear chain Sr_2CuO_3 in low temperature. This phenomenon is not observed in the zigzag chain $SrCuO_2$, even though the three dimensional ordering temperature ~ 2 K of the zigzag chain $SrCuO_2$ is less than the temperature ~ 5 K of the linear chain Sr_2CuO_3. We would like to point out the fact that the renormalization group flow (11) has a certain parameter region of the ferromagnetic interchain coupling in which

$$\chi(T) \sim \frac{1}{\pi v}\,(1 - \lambda_1(1/T)/4 - \lambda_2(1/T)/4) \tag{21}$$

does not decrease so drastically in the low temperature region. It might be possible to understand the difference of the low temperature behavior of the susceptibility between the linear chain and the zigzag chain in terms of RG flow.

The universal scale reduction in the zigzag chain is interesting as a rare example from the view point of the hierarchy problem in the theoretical physics. A string theory as an ultimate unified theory of every thing in particle physics needs a natural explanation of the strong suppression of the energy scale. Such a unified theory should explain the hierarchy of masses of the elementary excitations as well as the hierarchy of all interactions. We should obtain the mass of the light particle as a reduction from the Planck mass $\sim 10^{28}$ eV, for example, the neutrino mass ~ 1 eV, the electron mass $\sim 10^6$ eV, the muon mass $\sim 10^8$ eV, the Z boson mass $\sim 10^{11}$ eV. To understand this mass hierarchy problem in a unified theory, we have to answer the following two questions. First, why do several energy scales appear depending on the species of the fundamental excitations ? Second, how is the energy scale reduced with very large rate? Normally, the small mass of fermionic excitations is considered as the result of a small chiral symmetry breaking. For the small mass of bosonic excitations, the super symmetry[18] or composite approaches may work. For the too large reduction rate, however, we have to tune the coupling constants to the corresponding small region. This is possible, but unnatural. These two questions seem too difficult to resolve. In condensed matter physics, however, we have interesting examples to consider these problems. For the first question, we can refer the charge-spin separation in the Tomonaga-Luttinger liquid. Even though this model has the explicit symmetry breaking, the two fundamental excitations have different energy scales in the long distance physics. For the second question, a stable large scale reduction occurs in the ferromagnetic zigzag chain model without any fine-tuning of the coupling constant J_1. In the almost models, as in the antiferromagnetic zigzag chain model, however, the fine tuning of the coupling constant to the small region realizes the large scale reduction, even if the spontaneous chiral symmetry breaking works there. The experience on these examples may be useful to find a solution of the mass hierarchy problem.

This model should be studied still in more accurate approaches than the one loop renormalization group.

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